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1973 J. Phys. A: Math. Nucl. Gen. 6 1679

(http://iopscience.iop.org/0301-0015/6/11/007)

J. Phys. A: Math., Nucl. Gen., Vol. 6, November 1973. Printed in Great Britain. © 1973

XY model and algebraic methods

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Received 11 May 1973

Abstract. A systematic study of properties of a set of operators leads to a new approach to solve the XY model. The algebraic solution refers only to a subalgebra of spin operators. Though the connection with the algebra of related fermion operators is apparent, these need not be referred to. As a by-product, Onsager's method for the Ising model results in a very simple form, and its relationship to the approach of Schultz, Mattis and Lieb becomes transparent.

1. Properties of a simple set of operators

The XY model in one dimension is of a very simple structure (Lieb *et al* 1961, Katsura 1962). The eigenfunctions and eigenvalues of the symmetric hamiltonian are known as a special case of the Bethe solution of a more general class of Heisenberg hamiltonians, and at the same time a direct algebraic solution is known in terms of fermion quasi-particle operators. The significance of the model is partly in its role as a special case of more general Heisenberg hamiltonians, partly through its direct connection with simple models of statistical mechanics.

A new way to look at the solution of this model may shed some light on possible extensions, and may pave the way also towards an understanding of the model in several dimensions. This was the motivation of the present algebraic approach in which a reference to fermion operators is made only to establish the connection with known methods, and all the relationships can be and have been established without introducing them. These fermion operators may not be of much help in more than one dimension.

The hamiltonian of the XY model, with a constant asymmetry parameter Γ, can be written in the form

$$\mathscr{H} = -\frac{1+\Gamma}{2}A_1^{xx}-\frac{1-\Gamma}{2}A_1^{yy}, \tag{1a}$$

with

$$A_1^{xx} = \tfrac{1}{2}\sum_{j=1}^{N}\sigma_j^x\sigma_{j+1}^x, \tag{1b}$$

$$A_1^{yy} = \tfrac{1}{2}\sum_{j=1}^{N}\sigma_j^y\sigma_{j+1}^y. \tag{1c}$$

The Pauli matrices $\sigma_j^x, \sigma_j^y, \sigma_j^z$ are defined on the sites j of a ring, where $j = 1, 2, \ldots, N$, with periodicity conditions $\sigma_{j+N}^x = \sigma_j^x, \sigma_{j+N}^y = \sigma_j^y, \sigma_{j+N}^z = \sigma_j^z$.

If one forms the commutator $[A_1^{xx}, A_1^{yy}]$, then forms the commutator of this with A_1^{xx} and with A_1^{yy} and continues the process, one is led to consider the set of operators

$$A_l^{xx} = \tfrac{1}{2}\sum_{j=1}^{N} \sigma_j^x \sigma_{j+1}^z \dots \sigma_{j+l-1}^z \sigma_{j+l}^x \tag{2a}$$

$$A_l^{yy} = \tfrac{1}{2}\sum_{j=1}^{N} \sigma_j^y \sigma_{j+1}^z \dots \sigma_{j+l-1}^z \sigma_{j+l}^y \tag{2b}$$

$$A_l^{xy} = \tfrac{1}{2}\sum_{j=1}^{N} \sigma_j^x \sigma_{j+1}^z \dots \sigma_{j+l-1}^z \sigma_{j+l}^y \tag{2c}$$

$$A_l^{yx} = \tfrac{1}{2}\sum_{j=1}^{N} \sigma_j^y \sigma_{j+1}^z \dots \sigma_{j+l-1}^z \sigma_{j+l}^x . \tag{2d}$$

For $l = 1$, the expressions (2*a*), (2*b*) give (1*b*), (1*c*). The commutator algebra of these operators is closed. It is reduced by introducing the symmetric and antisymmetric combinations

$$A_l^{(xy)} = A_l^{xy} + A_l^{yx}, \tag{2e}$$

$$A_l^{\{xy\}} = A_l^{xy} - A_l^{yx}, \tag{2f}$$

since $A_l^{\{xy\}}$ can be shown to commute with all $A_{l'}^{xx}, A_{l'}^{yy}, A_{l'}^{(xy)}$.

In terms of the spin raising and lowering operators

$$\sigma_j^{\pm} = \tfrac{1}{2}(\sigma_j^x \pm \mathrm{i}\sigma_j^y)$$

one can alternatively consider the quantities

$$A_l^{+-} = \sum_{j=1}^{N} \sigma_j^+ \sigma_{j+1}^z \dots \sigma_{j+l-1}^z \sigma_{j+l}^- \tag{3a}$$

$$A_l^{-+} = \sum_{j=1}^{N} \sigma_j^- \sigma_{j+1}^z \dots \sigma_{j+l-1}^z \sigma_{j+l}^+ \tag{3b}$$

$$A_l^{++} = \sum_{j=1}^{N} \sigma_j^+ \sigma_{j+1}^z \dots \sigma_{j+l-1}^z \sigma_{j+l}^+ \tag{3c}$$

$$A_l^{--} = \sum_{j=1}^{N} \sigma_j^- \sigma_{j+1}^z \dots \sigma_{j+l-1}^z \sigma_{j+l}^- \tag{3d}$$

for which

$$(A_l^{+-})^\dagger = A_l^{+-}, \qquad (A_l^{-+})^\dagger = A_l^{-+}, \tag{3e}$$

$$(A_l^{++})^\dagger = A_l^{--}. \tag{3f}$$

The two sets of operators are related by the equations

$$A_l^{xx} + A_l^{yy} = A_l^{+-} + A_l^{-+}, \tag{4a}$$

$$A_l^{xx} - A_l^{yy} = A_l^{++} + A_l^{--}, \tag{4b}$$

$$A_l^{(xy)} = -\mathrm{i}(A_l^{++} - A_l^{--}), \tag{4c}$$

$$A_l^{\{xy\}} = \mathrm{i}(A_l^{+-} - A_l^{-+}). \tag{4d}$$

If one defines the operator

$$U = \sigma_1^z \sigma_2^z \dots \sigma_N^z, \qquad U^2 = 1, \tag{5a}$$

this commutes with σ_j^z and anticommutes with $\sigma_j^{\pm}$, and consequently it commutes with the previous sets of operators. It also connects operators with index l and $l+N$ in a simple way, according to the relationships

$$A_{l+N}^{+-} = -A_l^{+-}U, \qquad A_{l+N}^{-+} = -A_l^{-+}U, \tag{5b}$$

$$A_{l+N}^{++} = -A_l^{++}U, \qquad A_{l+N}^{--} = -A_l^{--}U. \tag{5c}$$

From the definitions one has for instance

$$A_{l+N}^{+-} = \sum_{j=1}^{N} \sigma_j^+ \sigma_{j+1}^z \dots \sigma_{j+l-1}^z (\sigma_{j+l}^z \dots \sigma_{j+l+N-1}^z) \sigma_{l+j+N}^-. \tag{5d}$$

Since $\sigma_{j+l}^z \dots \sigma_{j+l+N-1}^z = U$, and because of the periodicity conditions $\sigma_{l+j+N}^- = \sigma_{l+j}^-$, the relationship $A_{l+N}^{+-} = -A_l^{+-}U$ follows from $U\sigma_{l+j}^- + \sigma_{l+j}^- U = 0$.

On the other hand, the sets of operators considered are periodic with period $2N$, that is they are invariant with respect to replacing l by $l+2N$. With $U^2 = 1$, the relationships (5*b*), (5*c*) give immediately

$$A_{l+2N}^{+-} = -A_{l+N}^{+-}U = A_l^{+-}, \qquad A_{l+2N}^{-+} = A_l^{-+}, \tag{5e}$$

$$A_{l+2N}^{++} = A_l^{++}, \qquad A_{l+2N}^{--} = A_l^{--}. \tag{5f}$$

The definitions (3*a*)–(3*d*) of these operators A_l imply $l > 0$, but through the periodicity with $2N$ one can extend the definitions to $l \leqslant 0$, by writing

$$A_{-l} = A_{2N-l}, \qquad A_0 = A_{2N}. \tag{6a}$$

For l not equal to zero or to a multiple of N, one finds the identities

$$A_{-l}^{+-} = A_l^{-+}, \qquad A_{-l}^{-+} = A_l^{+-}, \tag{6b}$$

$$A_{-l}^{++} = -A_l^{++}, \qquad A_{-l}^{--} = -A_l^{--}. \tag{6c}$$

To show the first of these, write

$$A_{-l}^{+-} = A_{2N-l}^{+-} = \sum_{j=1}^{N} \sigma_j^+ (\sigma_{j+1}^z \dots \sigma_{j+2N-l-1}^z) \sigma_{j+2N-l}^-, \tag{6d}$$

with $U^2 = \sigma_{j+1}^z \dots \sigma_{j+2N}^z = 1$, one has

$$\begin{aligned}\sigma_{j+1}^z \dots \sigma_{j+2N-l-1}^z &= \sigma_{j+1}^z \dots \sigma_{j+2N-l-1}^z U^2 = \sigma_{j+2N-l}^z \sigma_{j+2N-l+1}^z \dots \sigma_{j+2N}^z \\ &= \sigma_{j-l}^z \sigma_{j-l+1}^z \dots \sigma_j^z \end{aligned} \tag{6e}$$

which gives

$$A_{-l}^{+-} = \sum_{j=1}^{N} \sigma_{j-l}^- \sigma_{j-l+1}^z \dots \sigma_{j-1}^z \sigma_j^+, \tag{6f}$$

where after substitution, the last factor $\sigma_{j+2N-l}^- = \sigma_{j-l}^-$ of the explicitly indicated term of (6*d*) has been commuted through all factors except σ_{j-l}^z with which it was combined according to $\sigma_{j-l}^z \sigma_{j-l}^- = -\sigma_{j-l}^-$. Similarly the first factor σ_j^+ has been commuted through all the other factors and combined with σ_j^z according to $\sigma_j^+ \sigma_j^z = -\sigma_j^+$. The expression (6*f*) differs from A_l^{-+} only through a relabelling of its terms.

For $l=0$, one finds the identities

$$A_0^{+-} = -\sum_{j=1}^{N} \tfrac{1}{2}(1+\sigma_j^z), \qquad A_0^{-+} = \sum_{j=1}^{N} \tfrac{1}{2}(1-\sigma_j^z), \tag{7a}$$

$$A_0^{++} = 0, \qquad A_0^{--} = 0. \tag{7b}$$

To obtain the expression of A_0^{+-} write

$$A_0^{+-} = A_{2N}^{+-} = \sum_{j=1}^{N} \sigma_j^+ (U^2 \sigma_{j+2N}^z)\sigma_{j+2N}^-. \tag{7c}$$

With $U^2=1$, and $\sigma_j^+\sigma_j^z\sigma_j^- = -\sigma_j^z\sigma_j^+\sigma_j^- = -\frac{1}{2}(1+\sigma_j^z)$ the form (7*a*) of A_0^{+-} follows. Related expressions for $l=N$ are given through the relationships $A_N = -A_0 U$.

In forming the commutators of the set of operators (3*a*), (3*d*), one finds

$$[A_{l'}^{+-}, A_l^{+-}] = 0, \qquad [A_{l'}^{-+}, A_l^{-+}] = 0, \tag{8a}$$

$$[A_{l'}^{+-}, A_l^{-+}] = 0, \tag{8b}$$

$$[A_{l'}^{++}, A_l^{++}] = 0, \qquad [A_{l'}^{--}, A_l^{--}] = 0. \tag{8c}$$

As already mentioned, $A_{l'}^{\{xy\}} = \mathrm{i}(A_{l'}^{+-} - A_{l'}^{-+})$ commutes with all the other elements, and one has

$$[A_{l'}^{\{xy\}}, \tfrac{1}{2}(A_l^{+-} + A_l^{-+})] = 0, \tag{8d}$$

$$[A_{l'}^{\{xy\}}, A_l^{++}] = 0, \qquad [A_{l'}^{\{xy\}}, A_l^{--}] = 0. \tag{8e}$$

For the remaining commutators one obtains

$$[\tfrac{1}{2}(A_{l'}^{+-} + A_{l'}^{-+}), A_l^{++}] = -A_{l+l'}^{++} - A_{l-l'}^{++}, \tag{8f}$$

$$[\tfrac{1}{2}(A_{l'}^{+-} + A_{l'}^{-+}), A_l^{--}] = A_{l+l'}^{--} + A_{l-l'}^{--}, \tag{8g}$$

$$[A_{l'}^{++}, A_l^{--}] = A_{l+l'}^{+-} + A_{l+l'}^{-+} - A_{l-l'}^{+-} - A_{l-l'}^{-+}, \tag{8h}$$

which shows that the commutator algebra of $A_l^{+-}, A_l^{-+}, A_l^{++}, A_l^{--}$ is closed.

For the quantities A_l^{xx}, A_l^{yy}, $A_l^{(xy)}$, $A_l^{\{xy\}}$, the identities (6*b*), (6*c*) and (4*a*)–(4*d*) give

$$A_{-l}^{xx} = A_l^{yy}, \qquad A_{-l}^{yy} = A_l^{xx}, \tag{9a}$$

$$A_{-l}^{(xy)} = -A_l^{(xy)}, \qquad A_{-l}^{\{xy\}} = -A_l^{\{xy\}}. \tag{9b}$$

For $l=0$, from (7*a*), (7*b*) and (4*a*)–(4*d*) follows

$$A_0^{xx} = A_0^{yy} = -\tfrac{1}{2}\sum_{j=1}^{N} \sigma_j^z, \tag{10a}$$

$$A_0^{(xy)} = 0, \qquad A_0^{\{xy\}} = -\mathrm{i}N. \tag{10b}$$

The commutation relations (8*f*)–(8*h*) give

$$[(A_{l'}^{xx} - A_{l'}^{yy}), (A_l^{xx} + A_l^{yy})] = 2\mathrm{i}(A_{l+l'}^{(xy)} - A_{l-l'}^{(xy)}), \tag{11a}$$

$$[(A_{l'}^{xx} + A_{l'}^{yy}), A_l^{(xy)}] = 2\mathrm{i}((A_{l+l'}^{xx} - A_{l+l'}^{yy}) + (A_{l-l'}^{xx} - A_{l-l'}^{yy})), \tag{11b}$$

$$[A_{l'}^{(xy)}, (A_l^{xx} - A_l^{yy})] = 2\mathrm{i}(-(A_{l+l'}^{xx} + A_{l+l'}^{yy}) + (A_{l-l'}^{xx} + A_{l-l'}^{yy})). \tag{11c}$$

First it will be the commutator algebra of this set of operators that will be exploited.

2. A new approach to solve the symmetric *XY* model

For $\Gamma = 0$, the hamiltonian (1*a*) of the *XY* model can be written in the form

$$\mathscr{H}^0 = -\tfrac{1}{2}(A_1^{+-} + A_1^{-+}) \tag{12a}$$

with

$$A_1^{+-} = \sum_{j=1}^{N} \sigma_j^+ \sigma_{j+1}^-, \qquad A_1^{-+} = \sum_{j=1}^{N} \sigma_j^- \sigma_{j+1}^+. \tag{12b}$$

The commutation relations (8*a*), (8*b*) show that the hamiltonian $\mathscr{H}^0$ commutes with the set of operators $A_l^{+-} + A_l^{-+}$ and $A_l^{+-} - A_l^{-+}$. From (8*f*) one obtains on the other hand

$$[\mathscr{H}^0, A_l^{++}] = A_{l+1}^{++} + A_{l-1}^{++}. \tag{12c}$$

This equation permits the construction of a special set of eigenstates of $\mathscr{H}^0$ by looking for operator solutions b of the equation

$$[\mathscr{H}^0, b] = \epsilon b. \tag{13a}$$

If ϕ is an eigenstate of $\mathscr{H}^0$, then $b\phi$ will be an eigenstate too, with an eigenvalue difference ϵ. A possible choice of ϕ is the state ϕ_0 for which $\sigma_j^- \phi_0 = 0$, $\mathscr{H}^0 \phi_0 = 0$.

Looking for solutions b which are linear combinations

$$b = \sum_{l=1}^{2N} \alpha_l A_l^{++} \tag{13b}$$

of the operators A_l^{++}, the commutator $[\mathscr{H}^0, b]$ obtains the form

$$[\mathscr{H}^0, b] = \sum_{l=1}^{2N} \alpha_l (A_{l+1}^{++} + A_{l-1}^{++}) = \sum_{l=1}^{2N} (\alpha_{l-1} + \alpha_{l+1}) A_l^{++}. \tag{13c}$$

Because of $A_{-l}^{++} = -A_l^{++}$, there are only $N-1$ linearly independent terms in (13*b*), but a sufficient condition to satisfy equation (13*a*) is to choose for the coefficients α_l solutions of the equation system

$$\alpha_{l-1} + \alpha_{l+1} = \epsilon \alpha_l, \qquad \text{for } l = 1, 2, \ldots, 2N. \tag{14a}$$

These equations are solved by $\alpha_l \sim \exp(\pm \mathrm{i}Kl)$, or any linear combination

$$c_1 \exp(\mathrm{i}Kl) + c_2 \exp(-\mathrm{i}Kl),$$

with eigenvalue

$$\epsilon = 2 \cos K. \tag{14b}$$

Since $A_{2N}^{++} = A_0^{++} = 0$, one can take $\alpha_0 = 0$ and choose c_1, c_2 to give

$$\alpha_l = \frac{1}{2N} \sin Kl \tag{14c}$$

where the normalization factor is arbitrary. The periodicity $A_{l+2N}^{++} = A_l^{++}$ of the operators can be reflected in the periodicity condition

$$\alpha_{l+2N} = \alpha_l \tag{14d}$$

of the coefficients which is satisfied if

$$\exp(\mathrm{i}2NK) = 1. \tag{14e}$$

This gives

$$K = \frac{2\pi}{N} m \qquad \text{with } m = \tfrac{1}{2}, 1, \tfrac{3}{2}, 2, \ldots, N. \tag{14f}$$

The operators

$$b_K = \frac{1}{2N} \sum_{l=1}^{2N} (\sin Kl) A_l^{++} \tag{15a}$$

satisfy the equations

$$[\mathscr{H}^0, b_K] = (2 \cot K) b_K. \tag{15b}$$

They commute with each other,

$$[b_K, b_{K'}] = 0 \tag{15c}$$

and one has

$$b_{-K} = -b_K. \tag{15d}$$

The sum (15*a*) can be inverted to give

$$A_l^{++} = \sum_K b_K \sin Kl \tag{16a}$$

which is summed over $2N$ values of K and is essentially a Fourier expansion of A_l^{++}. For the adjoint one can write similarly

$$A_l^{--} = \sum_K b_K^\dagger \sin Kl. \tag{16b}$$

For integer or half-integer values of m in (14*f*) one has $\sin K(l+N) = \pm \sin Kl$, and for the coefficients $\alpha_l(K) \sim \sin Kl$ in (13*b*), (15*a*) one has accordingly $\alpha_{l+N}(K) = \pm \alpha_l(K)$. If one combines this fact with $A_{l+N}^{++} = -A_l^{++} U$, the sum (15*a*) obtains a projection operator factor

$$\mathscr{P}_\pm = \tfrac{1}{2}(1 \pm U), \tag{17a}$$

with

$$b_K = \frac{1}{N} \sum_{l=1}^{N} (\sin Kl) A_l^{++} \mathscr{P}_\pm \qquad \text{for } m \begin{cases} \text{half-integer} \\ \text{integer} \end{cases} \tag{17b}$$

In the expansion of $A_l^{++}\mathscr{P}_+$ in terms of b_K, on the other hand, only K values with half-integer m appear, and in the expansion of $A_l^{++}\mathscr{P}_-$ only K's with integer m.

From the equations (15*b*) follows also

$$[\mathscr{H}^0, b_{K_1} \ldots b_{K_n}] = \left(2 \sum_{\alpha=1}^{N} \cos K_\alpha\right) b_{K_1} \ldots b_{K_n}. \tag{18}$$

As will be seen in the following section, the spectral decomposition of $\mathscr{H}^0$ can also be obtained in terms of the special set of operators considered.

3. Fermion operators and pseudo-spin algebra

Define

$$U_1 = \sigma_1^z, \qquad U_2 = \sigma_1^z\sigma_2^z, \qquad \ldots, \qquad U_j = \sigma_1^z\sigma_2^z, \ldots, \sigma_j^z, \tag{19a}$$

where

$$U_N = U, \tag{19b}$$

and

$$U_0 = U_{2N} = 1. \tag{19c}$$

In terms of these operators, the Wigner–Jordan transformation which relates fermion operators to $\sigma_j^\pm$ can be written in the simple form

$$a_j = \sigma_j^+ U_{j-1}, \qquad a_j^\dagger = U_{j-1}\sigma_j^-. \tag{20a}$$

The creation operators a_j and annihilation operators $a_j^\dagger$ satisfy fermion anti-commutation relations.

One can also define

$$a_{j+N} = a_j U, \qquad a_{j+N}^\dagger = U a_j^\dagger, \tag{20b}$$

so that

$$a_{j+2N} = a_j, \qquad a_{j+2N}^\dagger = a_j^\dagger. \tag{20c}$$

The operators (3*a*)–(3*d*) can be expressed in terms of $a_j, a_j^\dagger$ as bilinear expressions

$$A_l^{+-} = -\sum_{j=1}^{N} a_j a_{j+l}^\dagger, \qquad A_l^{-+} = \sum_{j=1}^{N} a_j^\dagger a_{j+l}, \tag{21a}$$

$$A_l^{++} = -\sum_{j=1}^{N} a_j a_{j+l}, \qquad A_l^{--} = \sum_{j=1}^{N} a_j^\dagger a_{j+l}^\dagger. \tag{21b}$$

Though a reference to the fermion operators makes some aspects of the algebra more transparent, the set of operators $A_l^{+-}, A_l^{-+}, A_l^{++}, A_l^{--}$ define a much smaller subalgebra than the algebra of fermion creation and annihilation operators and may be still useful in problems where an introduction of fermion operators is no longer advisable.

If one wants to refer to Fourier transforms a_K of the fermion operators with the usual normalization given by

$$a_j = \exp\left(\frac{\mathrm{i}\pi}{4}\right)\frac{1}{N^{1/2}}\sum_K{}' a_K \exp(-\mathrm{i}Kj) \tag{22a}$$

in the subspaces $\mathscr{P}_\pm = \frac{1}{2}(1 \pm U)$, where Σ_K' stands for a corresponding summation over half-integer or integer K values, one has to substitute (22*a*) in operator expressions with a factor $\mathscr{P}_\pm$. With the expression (21*b*) of A_l^{++} one obtains

$$\begin{aligned} A_l^{++}\mathscr{P}_\pm &= -\mathrm{i}\sum_K{}'\sum_{K'}{}' \exp(-\mathrm{i}K'l)\left(\frac{1}{N}\sum_{j=1}^{N}\exp[-\mathrm{i}(K+K')j]\right)a_K a_{K'} \\ &= -\mathrm{i}\sum_K{}' \exp(\mathrm{i}Kl)a_K a_{-K}. \end{aligned} \tag{22b}$$

The last expression results by writing $\delta_{K,-K'}$ for the parenthesis in the double sum and summing with respect to K'. If instead one first performs the summation with respect to K (and then omits the primes), the result is

$$A_l^{++}\mathscr{P}_{\pm} = \mathrm{i}\sum_K{}' \exp(-\mathrm{i}Kl) a_K a_{-K}. \tag{22c}$$

In averaging the two expressions one can write $\sin Kl$ for the coefficients. With $\mathscr{P}_+ + \mathscr{P}_- = 1$, and in introducing the pair operators

$$b_K = a_K a_{-K}, \qquad b_K^\dagger = a_{-K}^\dagger a_K^\dagger \tag{23a}$$

one has also

$$A_l^{++} = \sum_K b_K \sin Kl,$$

and similarly

$$A_l^{--} = \sum_K b_K^\dagger \sin Kl$$

which are the same expressions as (16*a*), (16*b*).

With the help of the number operators

$$n_K = a_K a_K^\dagger \tag{23b}$$

one can write similar expressions for A_l^{+-}, A_l^{-+}. One has

$$A_l^{+-} = -\sum_K \exp(-\mathrm{i}Kl) n_K, \qquad A_l^{-+} = \sum_K \exp(-\mathrm{i}Kl)(1-n_K), \tag{23c}$$

$$A_l^{++} = \mathrm{i}\sum_K \exp(-\mathrm{i}Kl) b_K, \qquad A_l^{--} = \mathrm{i}\sum_K \exp(-\mathrm{i}Kl) b_K^\dagger. \tag{23d}$$

The sum and difference of the two expressions (23*c*) can be written in the form

$$A_l^{+-} + A_l^{-+} = \sum_K m_K \cos Kl \tag{24a}$$

$$\mathrm{i}(A_l^{+-} - A_l^{-+}) = \sum_K \overline{m}_K \sin Kl \tag{24b}$$

with

$$m_K = 1 - n_K - n_{-K}, \qquad m_K = m_{-K}, \tag{24c}$$

$$\overline{m}_K = n_K - n_{-K}, \qquad \overline{m}_K = -\overline{m}_{-K}. \tag{24d}$$

The expressions for A_l^{++}, A_l^{--} can be inverted to obtain (15*a*) and its adjoint for b_K, $b_K^\dagger$, whereas from (24*a*), (24*b*) one obtains

$$m_K = \frac{1}{2N}\sum_{l=1}^{2N} (A_l^{+-} + A_l^{-+}) \cos Kl, \tag{25a}$$

$$\overline{m}_K = \frac{1}{2N}\sum_{l=1}^{2N} \mathrm{i}(A_l^{+-} - A_l^{-+}) \sin Kl. \tag{25b}$$

Through these relationships the commutator algebra of the operators A_l^{+-}, A_l^{-+},

A_l^{++}, A_l^{--} is reduced to irreducible parts b_K, $b_K^\dagger$, m_K and $\bar{m}_K$, any of which commutes with any operator $b_{K'}$, $b_{K'}^\dagger$, $m_{K'}$ and $\bar{m}_{K'}$ for $K \neq \pm K'$. $\bar{m}_K$ commutes with b_K, $b_K^\dagger$, m_K, as can be seen from the expression (25*b*) and the previous commutation relations.

At this stage, however, there is no need to restrict attention to the commutator algebra. For b_K, $b_K^\dagger$, n_K, n_{-K} one has the simple operator products

$$n_K b_K = b_K, \qquad b_K n_K = 0, \tag{26a}$$

$$n_{-K} b_K = b_K, \qquad b_K n_{-K} = 0, \tag{26b}$$

$$b_K b_K^\dagger = n_K n_{-K}, \qquad b_K^\dagger b_K = (1-n_K)(1-n_{-K}), \tag{26c}$$

$$b_K^2 = 0, \qquad (b_K^\dagger)^2 = 0, \qquad n_K^2 = n_K, \qquad n_{-K}^2 = n_{-K}. \tag{26d}$$

The operator

$$P_K = n_K n_{-K} + (1-n_K)(1-n_{-K}) \tag{27a}$$

is a projection operator,

$$P_K^2 = P_K, \tag{27b}$$

and one has

$$m_K^2 = P_K, \tag{27c}$$

$$b_K b_K^\dagger + b_K^\dagger b_K = P_K, \tag{27d}$$

whereas from (26*a*), (26*b*) one obtains

$$b_K m_K = b_K, \qquad m_K b_K = -b_K, \tag{28a}$$

$$m_K b_K^\dagger = b_K^\dagger, \qquad b_K^\dagger m_K = -b_K^\dagger. \tag{28b}$$

The commutation relations that follow from (28*a*), (28*b*) and (26*c*) are

$$[m_K, b_K] = -2b_K, \qquad [m_K, b_K^\dagger] = 2b_K^\dagger, \tag{28c}$$

$$[b_K^\dagger, b_K] = m_K, \tag{28d}$$

which are the commutation relations of pseudo-spin operators.

Because of $\frac{1}{2}(1+U)\frac{1}{2}(1-U) = 0$, the product of any operator $b_K, b_K^\dagger, m_K, \bar{m}_K$ with an operator $b_{K'}, b_{K'}^\dagger, m_{K'}, \bar{m}_{K'}$ is zero if $K = (2\pi/N)m$ corresponds to an integer m and K' to a half-integer m'.

With the expression (12*a*) of the hamiltonian of the symmetric XY model, and the decomposition (24*a*), one can write

$$\mathscr{H}^0 = -\tfrac{1}{2}\sum_K m_K \cos K. \tag{29a}$$

This expression is equivalent to the spectral decomposition of $\mathscr{H}^0$, and the eigenvalues can be read off in a simple way. The operators $m_{K=0}$, $m_{K=\pi}$ appear only once in the sum (29*a*), the other m_K twice because of $m_K = m_{-K}$.

If one wants to give a complete set of eigenvectors of $\mathscr{H}^0$, one has to refer to operator products of the form $a_{K_1} \dots a_{K_n}$. The projection operators $P_{K_1 \dots K_n}$ of these states can be expressed, however, with elements of the subalgebra and are of the form

$$P_{K_1 \dots K_n} = \mathscr{P}_\pm \prod_{K=K_1\dots K_n}' n_K \prod_{\bar{K} \neq K_1 \dots K_n}' (1-n_{\bar{K}}). \tag{29b}$$

They commute with $\mathscr{H}^0$, and satisfy equations of the form

$$\mathscr{H}^0 P_{K_1 \ldots K_n} = \epsilon_{K_1 \ldots K_n} P_{K_1 \ldots K_n}, \tag{29c}$$

where the eigenvalues $\epsilon_{K_1 \ldots K_n}$ are given by the expression (29*a*).

In referring to these projection operators and determining energy values, it is not only the commutator algebra of A_l^{+-}, A_l^{-+}, A_l^{++}, A_l^{--} that is referred to but the full algebra of these quantities introduced through their operator products. After a reduction to the quantities $m_K, \bar{m}_K, b_K, b_K^\dagger$ with the help of the commutator algebra, the operations within this algebra are, however, rather simple.

4. Asymmetric *XY* model

The hamiltonian (1*a*) of the asymmetric *XY* model can be written in the form

$$\mathscr{H} = -\tfrac{1}{2}[(A_1^{+-} + A_1^{-+}) + \Gamma(A_1^{++} + A_1^{--})]. \tag{30a}$$

The eigenvalues and special pair solutions can be determined in j space by similar commutator methods as for $\Gamma = 0$. It is simpler, however, to use directly the decompositions (24*a*), (16*a*), (16*b*) and write

$$\mathscr{H} = \tfrac{1}{2} \sum_K \mathscr{H}_K \tag{30b}$$

with

$$\mathscr{H}_K = v_K m_K + \mu_K (b_K + b_K^\dagger), \tag{30c}$$

$$v_K = \cos K, \qquad \mu_K = \Gamma \sin K. \tag{30d}$$

Because of $m_K = m_{-K}$, $b_K = -b_{-K}$, $v_K = v_{-K}$, $\mu_K = -\mu_{-K}$ one has $\mathscr{H}_K = \mathscr{H}_{-K}$, and for $K \neq 0, \pi$ the terms of (30*b*) are pairwise the same which leads to a cancellation of the factor $\frac{1}{2}$.

The terms $\mathscr{H}_K$ of $\mathscr{H}$ commute, and each can be diagonalized by writing

$$E_K = (v_K^2 + \mu_K^2)^{1/2}, \tag{31a}$$

$$\cos \delta_K = \frac{v_K}{E_K}, \qquad \sin \delta_K = \frac{\mu_K}{E_K}, \tag{31b}$$

and introducing

$$\mathscr{M}_K = m_K \cos \delta_K + (b_K + b_K^\dagger) \sin \delta_K, \tag{31c}$$

$$B_K + B_K^\dagger = -m_K \sin \delta_K + (b_K + b_K^\dagger) \cos \delta_K, \tag{31d}$$

$$B_K - B_K^\dagger = b_K - b_K^\dagger, \tag{31e}$$

$$\bar{\mathscr{M}}_K = \bar{m}_K, \tag{31f}$$

which represents a simple rotation of the pseudo-spin operators. In the same way as $m_K, \bar{m}_K$ are related to the number operators n_K, n_{-K}, the operators $\mathscr{M}_K, \bar{\mathscr{M}}_K$ define $\mathscr{N}_K$, $\mathscr{N}_{-K}$ through the equations

$$\mathscr{M}_K = 1 - \mathscr{N}_K - \mathscr{N}_{-K}, \qquad \bar{\mathscr{M}}_K = \mathscr{N}_K - \mathscr{N}_{-K}. \tag{31g}$$

These satisfy $\mathcal{N}_K^2 = \mathcal{N}_K$, $\mathcal{N}_{-K}^2 = \mathcal{N}_{-K}$ and are fermion quasi-particle number operators. One has

$$B_K^2 = 0, \qquad (B_K^\dagger)^2 = 0, \tag{32a}$$

$$B_K B_K^\dagger + B_K^\dagger B_K = P_K, \tag{32b}$$

$$\mathcal{M}_K^2 = P_K, \tag{32c}$$

and the relationships (28*a*)–(28*d*) are transformed into

$$B_K \mathcal{M}_K = B_K, \qquad \mathcal{M}_K B_K = -B_K, \tag{33a}$$

$$\mathcal{M}_K B_K^\dagger = B_K^\dagger, \qquad B_K^\dagger \mathcal{M}_K = -B_K^\dagger \tag{33b}$$

and

$$[\mathcal{M}_K, B_K] = -2B_K, \qquad [\mathcal{M}_K, B_K^\dagger] = 2B_K^\dagger \tag{33c}$$

$$[B_K^\dagger, B_K] = \mathcal{M}_K. \tag{33d}$$

According to (30*c*), (31*a*)–(31*c*) the term $\mathcal{H}_K$ of the hamiltonian can be written as

$$\mathcal{H}_K = E_K \mathcal{M}_K. \tag{34a}$$

This gives commutator equations

$$[\mathcal{H}, B_K] = 2E_K B_K, \tag{34b}$$

$$[\mathcal{H}, B_{K_1} \dots B_{K_n}] = \left(2 \sum_{\alpha=1}^{n} E_{K_\alpha}\right) B_{K_1} \dots B_{K_n} \tag{34c}$$

and gives eigenstates $B_{K_1} \dots B_{K_n} \phi_0$ in the subspace P_- in which for ϕ_0 one has

$$B_K^\dagger \phi_0 = 0 \tag{35a}$$

$$\mathcal{H} \phi_0 = W_0 \phi_0 \tag{35b}$$

with

$$W_0 = -\tfrac{1}{2} \sum_K{}' E_K. \tag{35c}$$

The form (30*b*), (34*a*) of the hamiltonian is equivalent to its spectral decomposition in terms of commuting projection operators. In order to obtain it there is no need to introduce the fermion quasi-particle operators ξ_K, $\xi_{-K}^\dagger$ given by the transformation

$$a_K = \xi_K \cos \tfrac{1}{2}\delta_K + \xi_{-K}^\dagger \sin \tfrac{1}{2}\delta_K \tag{36a}$$

$$a_{-K} = -\xi_K \sin \tfrac{1}{2}\delta_K + \xi_{-K}^\dagger \cos \tfrac{1}{2}\delta_K \tag{36b}$$

and by the adjoint equations. The relationships

$$B_K = \xi_K \xi_{-K}, \qquad B_K^\dagger = \xi_{-K}^\dagger \xi_K^\dagger, \tag{36c}$$

$$\mathcal{N}_K = \xi_K \xi_K^\dagger, \tag{36d}$$

make the content of the algebraic rules (32*a*)–(32*c*), (33*a*)–(33*d*) nonetheless more transparent. The construction of a complete set of eigenvectors of the hamiltonian would have to refer to products $\xi_{K_1} \dots \xi_{K_n}$ of these quasi-particle operators. The projection operators of the eigenstates can, however, be expressed in terms of the subalgebra considered, and are of the form

$$\mathscr{P}_{K_1\ldots K_n} = \mathscr{P}_{\pm} \prod_{K=K_1\ldots K_n}' \mathscr{N}_K \prod_{\bar{K}\neq K_1\ldots K_n}' (1-\mathscr{N}_{\bar{K}}). \tag{37}$$

5. Transfer matrix of the Ising model

The partition function of the two-dimensional Ising model (Onsager 1944, Newell and Montroll 1953, Schultz *et al* 1964) in the absence of a magnetic field, can be obtained by determining the largest eigenvalue of the transfer matrix

$$V = V_1^{1/2} V_2 V_1^{1/2} \tag{38a}$$

where one can write

$$V_1 = C \exp\left(\tilde{H} \sum_{j=1}^{N} \sigma_j^z\right), \tag{38b}$$

$$V_2 = \exp\left(H' \sum_{j=1}^{N} \sigma_j^x \sigma_{j+1}^x\right), \tag{38c}$$

with

$$C = (2 \sinh 2H)^{N/2}. \tag{38d}$$

Here $H = J/kT$, $H' = J'/kT$ where J, J' are the coupling constants of the Ising model, k is Boltzmann's constant, T the temperature and $\tilde{H}$ is determined by the equation $\sinh 2H \sinh 2\tilde{H} = 1$.

With the notation (10*a*) and (1*b*), one can write

$$V_1 = C \exp(-2\tilde{H} A_0^{xx}), \tag{38e}$$

$$V_2 = \exp(2H' A_1^{xx}). \tag{38f}$$

Apart from factors and notation, Onsager's starting point is to form the commutator algebra generated by A_0^{xx} and A_1^{xx}, and in this way he obtains the operators A_l^{xx} and $A_l^{\{xy\}}$. The quantities A_l^{yy} and $A_l^{\{xy\}}$ of the more systematic treatment of spin operators do not appear in his work, but according to the relationship (9*a*) the A_l^{yy} can be obtained from the quantities A_l^{xx}. The set $A_l^{\{xy\}}$ is needed here only to construct the projection operators of the eigenstates. In reducing the commutator or Lie algebra, Onsager arrives at pseudo-spin operators expressible in terms of $m_K, b_K, b_K^\dagger$.

According to (10*a*) one has $A_0^{xx} = A_0^{yy}$ and with the relationships (4*a*), (24*a*) one can write

$$2A_0^{xx} = \sum_K m_K. \tag{39a}$$

At the same time, with (4*a*), (4*b*), (24*a*), (16*a*) and (16*b*) one has

$$2A_1^{xx} = \sum_K (m_K \cos K + (b_K + b_K^\dagger) \sin K). \tag{39b}$$

This decomposes the transfer matrix into commuting factors

$$V(K) = \hat{V}_1^{1/2}(K) V_2(K) \hat{V}_1^{1/2}(K), \tag{40a}$$

with

$$\hat{V}_1^{1/2}(K) = \exp(-\tilde{H} m_K), \tag{40b}$$

$$V_2(K) = \exp\{2H'[m_K \cos K + (b_K + b_K^\dagger) \sin K]\}. \tag{40c}$$

The explicitly given expressions are those for $K \neq 0, \pi$ and for these an extra factor two in the exponents results from the identity of terms with K and $-K$ in (39*a*), (39*b*). In the two subspaces given by the projection operators $\mathscr{P}_\pm = \frac{1}{2}(1 \pm U)$, the transfer matrix $V_\pm = \mathscr{P}_\pm V$ is of the form

$$V_\pm = C \prod_K{}' V(K). \tag{40d}$$

One has $m_K^2 = P_K$ and the square of the operator factor of $2H'$ in (40*c*) is also equal to the projection operator P_K. One can write (40*b*), (40*c*) accordingly as

$$\hat{V}_1^{1/2}(K) = (1 - P_K) + P_K(\cosh \tilde{H} - m_K \sinh \tilde{H}) \tag{41a}$$

$$V_2(K) = (1 - P_K) + P_K\{\cosh 2H' + [m_K \cos K + (b_K + b_K^\dagger) \sin K] \sinh 2H'\} \tag{41b}$$

In forming the product (40*a*), only the simple multiplication rules (27*b*)–(27*d*), (28*a*), (28*b*) are to be used to obtain

$$V(K) = (1 - P_K) + P_K[\hat{F}_K + \hat{\nu}_K m_K + \hat{\mu}_K(b_K + b_K^\dagger)] \tag{42a}$$

with

$$\hat{\nu}_K = -\sinh 2\tilde{H} \cosh 2H' + \cosh 2\tilde{H} \sinh 2H' \cos K, \tag{42b}$$

$$\hat{\mu}_K = \sinh 2H' \sin K, \tag{42c}$$

$$\hat{F}_K = \cosh 2\tilde{H} \cosh 2H' - \sinh 2\tilde{H} \sinh 2H' \cos K. \tag{42d}$$

As in (31*a*), (31*b*), one can write

$$\hat{E}_K = (\hat{\nu}_K^2 + \hat{\mu}_K^2)^{1/2} \tag{43a}$$

$$\cos \hat{\delta}_K = \frac{\hat{\nu}_K}{\hat{E}_K}, \qquad \sin \hat{\delta}_K = \frac{\hat{\mu}_K}{\hat{E}_K} \tag{43b}$$

and perform a transformation (31*c*)–(31*f*) which results in

$$V(K) = (1 - P_K) + P_K(\hat{F}_K + \hat{E}_K \mathscr{M}_K). \tag{43c}$$

Because of the identity $\hat{F}_K^2 - \hat{E}_K^2 = 1$, which follows from the definitions, one can also write

$$\hat{F}_K = \cosh \epsilon_K, \qquad \hat{E}_K = -\sinh \epsilon_K \tag{43d}$$

and

$$V(K) = (1 - P_K) + P_K \exp(-\epsilon_K \mathscr{M}_K) = \exp(-\epsilon_K \mathscr{M}_K). \tag{43e}$$

The simple form of the transfer matrix obtained from (43*c*) or (43*e*) is equivalent to its spectral decomposition in terms of projection operators of the form (37).

Onsager's method can be seen in this way to involve only rather elementary considerations. The close connection with the method of Schultz *et al* is evident from the previous discussion of the relationship between fermion operators and pseudo-spin algebra. The presentation given here is strongly based on their insights concerning the connection of the diagonalization of V with the quasi-particle transformation.

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